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# A NECESSARY CONDITION FOR $\underset{a=b}{A * B}$ TO BE LERF

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ABSTRACT

This paper examines some necessary conditions for the product $\underset{a=b}{A * B}$ to be LERF.

A group $G$ is LERF (locally extended residually finite) if for any finitely generated subgroup $C$ of $G$ and for any element $g \in G \setminus C$ there exists a finite index subgroup $G_0$ of $G$ which contains $C$ but not $g$.

A free group is LERF [Hall] and a free product of free groups with cyclic amalgamation is LERF [B-B-S].

In [Gi] sufficient conditions on a LERF group $B$ and an element $b \in B$ of infinite order were given such that the group $G = \underset{a=b}{A * B}$ is LERF for any LERF group $A$ and any element $a \in A$ of infinite order.

On the other hand, the examples show [Ri] that a free product of LERF groups with cyclic amalgamation need not be LERF.

In this note we study some necessary conditions for such a product to be LERF. The comparison shows that there is still a considerable gap between necessary and sufficient conditions which requires additional investigation.

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THEOREM. *Let $D$ be a free abelian group with the basis $d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8$. Let $A = D \rtimes \langle a \rangle$, where $a^{-1}d_1a = d_1d_2$, $a^{-1}d_2a = d_2$, $a^{-1}d_3a = d_4$, $a^{-1}d_4a = d_3$, $a^{-1}d_5a = d_6$, $a^{-1}d_6a = d_5$, $a^{-1}d_7a = d_7$, $a^{-1}d_8a = d_8$.*

*Let $B$ be any group containing an element $b$ of infinite order. If $B$ contains an element $c$ such that $bc = cb$ and $c \notin \langle b \rangle$, then $G = \underset{a=b}{A * B}$ is not LERF, though $A$ is LERF.*

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PROOF.

$D \rtimes \langle a^2 \rangle$ is finitely generated nilpotent and of index 2 in $A$, so $A$ is LERF.

Let us show that $G$ is not LERF.

Consider the subgroup $C = \langle ad_2, cd_1, d_7d_3cd_3, d_4cd_4, d_8d_5cd_5, d_6cd_6, ad_7d_8 \rangle$. We show that:

(a) $a \notin C$;

(b) for any subgroup $H$ containing $C$, either $a \in H$ or $\langle a \rangle \cap H = \{1\}$.

Hence $G$ is not LERF because the element $a$ belongs to any finite index subgroup of $G$ containing $C$.

In order to prove (a) let us determine $C \cap A$.

Let $w_1, w_2 \in \{cd_1, d_3d_7cd_3, d_4cd_4, d_5d_8cd_5, d_6cd_6\}$, $\epsilon = \pm 1$ and $\delta = \pm 1$. Then $w_1^\epsilon = u_1^{-1}c^\epsilon v_1$ and $w_2^\delta = u_2^{-1}c^\delta v_2$, where $u_1, u_2, v_1, v_2 \in \{1, d_1, d_7^{-1}d_3^{-1}, d_3, d_4^{-1}, d_4, d_8^{-1}d_5^{-1}, d_5, d_6^{-1}, d_6\}$.

Note that the elements $ad_2$ and $ad_7d_8$ commute.

CLAIM. *For any $k, l \in Z$, $h_{k,l} = v_1(ad_2)^k(ad_7d_8)^l u_2^{-1} \notin \langle a \rangle$ unless $k = l = 0$, $w_1 = w_2$ and $\epsilon = -\delta$.*

We prove the claim in several steps.

(1) Let $D_0 = \langle d_3, d_4, d_5, d_6, d_7, d_8 \rangle$, $\bar{A} = A/D_0$. We have $d_1ad_1^{-1} = ad_2$, therefore, in $\bar{A}$, if $\bar{v}_1$ and $\bar{u}_2$ are in $\{\bar{1}, \bar{d}_1\}$, then $\bar{v}_1(\bar{a}\bar{d}_2)^k\bar{a}^l\bar{u}_2^{-1} \notin \langle \bar{a} \rangle$ unless $\bar{v}_1 = \bar{u}_2 = d_1$ and $2k + l = 0$ or $\bar{v}_1 = \bar{u}_2 = \bar{1}$ and $k = 0$.

If $v_1 = u_2 = d_1$, then in view of $d_1ad_1^{-1} = ad_2$ it follows that $h_{k,l} = v_1(ad_2)^k(ad_7d_8)^l u_2^{-1} = d_1(a^{k+l}d_2^kd_7^ld_8^l)d_1^{-1} = a^{k+l}d_2^{2k+l}d_7^ld_8^l$, so $h_{k,l} \in \langle a \rangle$ implies that $l = 0$ and $2k + l = 0$, hence $k = l = 0$, while $w_1 = w_2 = cd_1$, $\epsilon = 1$ and $\delta = -1$.

(2) If $v_1 = d_1$ and $u_2 \neq d_1$ or $v_1 \neq d_1$ and $u_2 = d_1$, then for $D_1 = \langle d_2, d_3, d_4, d_5, d_6, d_7, d_8 \rangle$ we have $h_{k,l} \equiv a^{k+l} \cdot d_1^{\pm 1} (\bmod D_1)$, so $h_{k,l} \notin \langle a \rangle$.

(3) If $v_1 \neq d_1$ and $u_2 \neq d_1$, then for $D_2 = \langle d_3, d_4, d_5, d_6, d_7, d_8 \rangle$ we have $h_{k,l} \equiv a^{k+l}d_2^k (\bmod D_2)$, so if $h_{k,l} \in \langle a \rangle$, then $k = 0$.

Now we assume that $v_1 \neq d_1$, $u_2 \neq d_1$ and $k = 0$.

(4) If $u_2 = v_1 = d_7^{-1}d_3^{-1}$ or $v_1 = u_2 = d_8^{-1}d_5^{-1}$, then for $D_3 = \langle d_1, d_2, d_3, d_4, d_5, d_6 \rangle$ we have that $h_{0,l} \equiv a^ld_7^ld_8^l (\bmod D_3)$, so $h_{0,l} \in \langle a \rangle$ implies $l = 0$, $w_1 = w_2 = d_3d_7cd_3$ or $w_1 = w_2 = d_5d_8cd_5$ and $\epsilon = -\delta$.

(5) In all the other cases $h_{0,l} \equiv a^ld_7^{l+\eta_1}d_8^{l+\eta_2} (\bmod D_3)$, where $\eta_1 = -1$ if $v_1 = d_7^{-1}d_3^{-1}$, $\eta_1 = 1$ is $u_2 = d_7^{-1}d_3^{-1}$ and $\eta_1 = 0$ otherwise, and $\eta_2 = 1$ if $u_2 = d_8^{-1}d_5^{-1}$, $\eta_2 = -1$ if $v_1 = d_8^{-1}d_5^{-1}$ and $\eta_2 = 0$ otherwise. Hence $h_{0,l} \in \langle a \rangle$ implies $l + \eta_1 = l + \eta_2 = 0$, so $\eta_2 = \eta_1 = 0$ and $l = 0$.

For $v_1, u_2 \in \langle 1, d_7^{-1}d_3^{-1}, d_4^{-1}, d_4, d_8^{-1}d_5^{-1}, d_5, d_6^{-1}, d_6\rangle$ we have that $h_{0,0} = v_1u_2^{-1} \in \langle a\rangle$ only when $v_1 = u_2$, so $w_1 = w_2$ and $\epsilon = -\delta$, as required.

The claim is proved, so by the normal form theorem in the amalgamated products $C \cap A = \langle ad_2, ad_7d_8\rangle$, hence $a \notin C \cap A$. □

Let $H$ be a subgroup of $G$ containing $C$ and $a^n$.

(b) is a consequence of (c) and (d).

(c) If $n = 2m + 1$, then $a \in H$.

Indeed, for $n$ odd, $a^{-n}d_3a^n = d_4$, $a^{-n}d_4a^n = d_3$, $a^{-n}d_5a^n = d_6$ and $a^{-n}d_6a^n = d_5$, so $a^{-n}(d_3d_7cd_3)a^n(d_4cd_4)^{-1} = d_7d_4cd_4 \cdot (d_4cd_4)^{-1} = d_7 \in H$, and $a^{-n}(d_5d_8cd_5)a^n(d_6cd_6)^{-1} = d_8d_6cd_6 \cdot (d_6cd_6)^{-1} = d_8 \in H$, hence $ad_7d_8 \cdot d_8^{-1}d_7^{-1} \in H$.

(d) If $n = 2m$, then $a^m \in H$.

Indeed, $d_1ad_1^{-1} = ad_2$, so $(cd_1)(a^{2m}(ad_2)^{-m})(cd_1)^{-1} = c(d_1(a^md_2^{-m})d_1^{-1})c^{-1} = ca^mc^{-1} = a^m \in H$. □

Remark 1. Taking $B = \langle c\rangle$, $b = c^k$, we obtain that adjunction of roots need not preserve the property LERF.

Remark 2. If $B$ contains an element $c_0$ such that $bc_0 = c_0b^{-1}$, then considering the subgroup $C_0 = \langle ad_2, c_0d_1, d_7d_3c_0d_3c_0^{-1}d_3, d_4c_0d_4c_0^{-1}d_4, d_8d_5c_0d_5c_0^{-1}d_5, d_6c_0d_6c_0^{-1}d_6, ad_7d_8\rangle$ we similarly show that $G = A \underset{a=b}{*} B$ is not LERF. □

Corollary. *Let $B$ be a LERF group and $b \in B$ be of infinite order. If for any LERF group $A$ and $a \in A$ of infinite order $A \underset{a=b}{*} B$ is LERF, then $N_B(b) = \langle b\rangle$.* □

## References

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